

A Genetic Algorithm for The Forwarding Index/Diameter of Graphs

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Abstract

The forwarding index/diameter are notions introduced to formalise problems arising in the forwarding of messages in communication networks. Formally speaking, a routing R in a graph G is a set of paths $\{R_{xy} : x, y \in V(G)\}$ where, for each ordered pair of vertices (x, y) , R_{xy} links x to y . The load $\xi(G, R, x)$ of a vertex x in the routing R is the number of paths of R for which x is an interior vertex. The forwarding index $\xi(G, R)$ of the pair (G, R) is defined as the maximum of the loads $\xi(G, R, x)$ taken over all vertices. The forwarding index of G , denoted by $\xi(G)$, is defined as the minimum of the forwarding indices $\xi(G, R)$ taken over all possible routings. The forwarding diameter $\mu(G, R)$ of the pair (G, R) is defined by

$$\mu(G, R) = \max_{(x,y)} \sum_{z \in R_{xy} - \{x,y\}} \xi(G, R, z)$$

and the forwarding diameter $\mu(G)$ of G as the minimum of $\mu(G, R)$ taken over all possible routings. The problem of the forwarding index (forwarding diameter) consists, given an integer m and a connected graph G , in finding a routing R such that $\xi(G, R) \leq m$ ($\mu(G, R) \leq m$). In this paper, we propose a naturally genetic coding and develop a simple genetic algorithm for the forwarding index/diameter problem. We show also experimental results of this genetic algorithm.

1 Introduction

Let $G(V, E)$ be a simple connected graph on n vertices with vertex set $V = V(G)$ and edge-set $E = E(G)$. An edge between two vertices x and y is denoted by xy and $d(x, y)$ denotes the distance between x and y in G , i.e. the length of a shortest path between x and y .

A *routing* of G is a set $R = \{R_{xy} : x, y \in V(G), x \neq y\}$ of elementary paths connecting each ordered pair (x, y) of vertices of G . Thus R_{xy} is not necessarily the same as R_{yx} . A *routing of shortest paths*, denoted by R_m , is a routing where all the paths are shortest paths.

Chung, Coffman, Reiman, and Simon, introduced in [2] the notion of forwarding index which has obvious practical significance in network design and is defined as follows. The *load* $\xi(G, R, x)$ of a vertex x in the routing R is defined as the number of paths of R for which x is an interior vertex.

The *vertex-forwarding index or forwarding index of the pair* (G, R) is defined as

$$\xi(G, R) = \max_{x \in V(G)} \xi(G, R, x).$$

The *forwarding index or vertex-forwarding index of G* [2], denoted by $\xi(G)$, is defined as the minimum of the forwarding indices $\xi(G, R)$ taken over all possible routings:

$$\xi(G) = \min_R \xi(G, R).$$

The minimum taken over all routings of shortest path is called *forwarding*

index by routing of shortest paths and denoted by $\xi_m(G)$:

$$\xi_m(G) = \min_{R_m} \xi(G, R_m).$$

Roughly speaking, the “best” routing is the routing that maximum of the load is minimum. The forwarding index appears to be a good measure of the maximum congestion of the nodes in a network, in which the message traffic is nearly equal for the distinct pairs of nodes. Fernández de la Vega, El Haddad, Barráez and Ordaz introduced in [5] the notion of forwarding diameter. The forwarding diameter appears to be more suited for the case where one wants to minimize the maximum message transmission delay.

For a given routing R , the *forwarding distances* between the vertices x and y is defined by

$$\mu_R(x, y) = \sum_{z \in R_{xy} - \{x, y\}} \xi(G, R, z).$$

The *forwarding diameter* $\mu(R, G)$ corresponding to the routing R is defined by

$$\mu(R, G) = \max_{x, y \in V(G)} \mu_R(x, y)$$

and the *forwarding diameter* $\mu(G)$ of G by

$$\mu(G) = \min_R \mu(R, G)$$

where the minimum is taken over all possible routings. The minimum taken over all routings of shortest paths is called *forwarding diameter by routing of shortest paths* and denoted by $\mu_m(G)$:

$$\mu_m(G) = \min_{R_m} \mu(R_m, G).$$

In what follows, we suppose without loss of generality that if (x, y) is an edge then $R(x, y) = xy$ (and $\mu_R(x, y) = 0$). The following bounds are well known.

Proposition 1.1 (Chung, Coffman, Reiman, Simon [2]) *Let G be a connected graph of order n . Then*

$$(i) \quad \frac{1}{n} \sum_u \sum_{v \neq u} (d(u, v) - 1) \leq \xi(G) \leq \xi_m(G) \leq (n-1)(n-2).$$

(ii) *The equalities*

$$\frac{1}{n} \sum_u \sum_{v \neq u} (d(u, v) - 1) \leq \xi(G) \leq \xi_m(G)$$

are true if and only if there exists a routing of shortest paths in G which loads all vertices equally.

Proposition 1.2 (Fernández, El Haddad, Barráez, Ordaz [5, 1]) *Let $G = (V(G), E(G))$ be a connected graph of order n such that $G \neq K_n$, (K_n complete graph of order n), then*

$$(i) \quad \frac{(\sum_u \sum_{v \neq u} (d(u, v) - 1))^2}{n(n(n-1) - 2 |E(G)|)} \leq \mu(G) \leq \mu_m(G) \leq (n-1)(n-2)^2.$$

(ii) *The equalities*

$$\frac{(\sum_u \sum_{v \neq u} (d(u, v) - 1))^2}{n(n(n-1) - 2 |E(G)|)} = \mu(G) = \mu_m(G)$$

are true if and only if there exists a routing of shortest paths in G which loads all vertices equally such that for every pair of non adjacent vertices (x, y) , $\mu_R(x, y) = \mu_R(G)$.

The computation of the forwarding index/diameter of a given graph is a very difficult combinatorial problem. For very few graphs the forwarding index/diameter or even good estimations of them are known [2, 3, 5, 1]. Formally, the forwarding index problem (forwarding diameter) consists, given an integer m and a connected graph G , in finding a routing R (R') such that $\xi(G, R) \leq m$ ($\mu(G, R') \leq m$). The forwarding index problem is NP-complete [7]. The complexity of the forwarding diameter problem is unknown in the general case. No algorithm is known for the two problems described above with the only exemption of the particular case of routings of shortest paths for graphs of diameter 2, where both problems are polynomials [4, 1].

2 Description of the algorithm and experimental results

In what follows, we propose a naturally genetic coding and a simple genetic algorithm. Let $p_1, p_2, \dots, p_{n(n-1)}$ be the set of the distinct pairs of vertices of the graph G . A fixed routing R we can represent by a chromosome of $n(n-1)$ genes, where the i^{th} gene is an elementary path between the pair of vertices p_i . Note that in this genetic coding, when we applied the usually

crossover operator on two chromosome we obtain another chromosome, which is also a routing.

Following [6], we denote by P_0 the initial population, $|P_0|$ the initial population size, P_n the n^{th} -generation, F the fitness function, f_i the value of the fitness function F in the i^{th} -individual of the population, p_{s_i} the probability that i^{th} -individual of the population is selected for reproduction, p_c the probability of crossover, and p_m the probability of mutation.

The fitness function is defined by

$$F(x) = \begin{cases} C_{max} - g(x) & \text{when } g(x) < C_{max} \\ 0 & \text{otherwise} \end{cases}$$

In the case of the forwarding index the value of the function g in the routing R is $\xi(G, R)$. In the initial population the value of $C_{max} = (n-1)(n-2)$, the upper bound given by Theorem 1.1, and in the other populations the value of C_{max} is equal to the maximum value of g in the initial population. The case of the forwarding diameter is similar, the value of the function g in the routing R is $\mu(G, R)$. In the initial population the value of $C_{max} = (n-1)(n-2)^2$, the upper bound given by Theorem 1.2, and in the other populations the value of C_{max} is equal to the maximum value of g in the initial population. The values of the other parameters are

$$|P| = 180$$

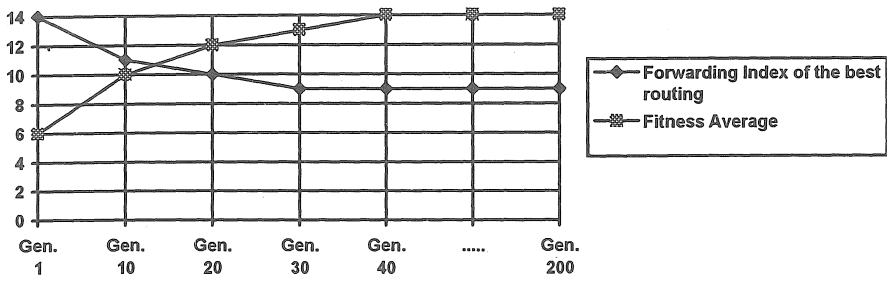
$$p_c = 0.8$$

$$p_m = 0.05$$

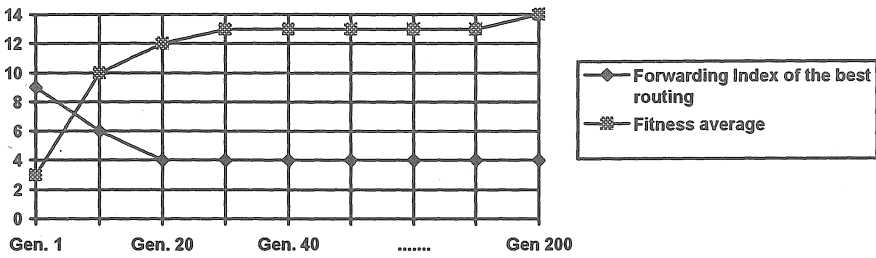
$$p_{s_i} = \frac{f_i}{\sum_i f_i}.$$

Note that the genetic algorithm in the two cases is almost the same, differing only in the fitness function.

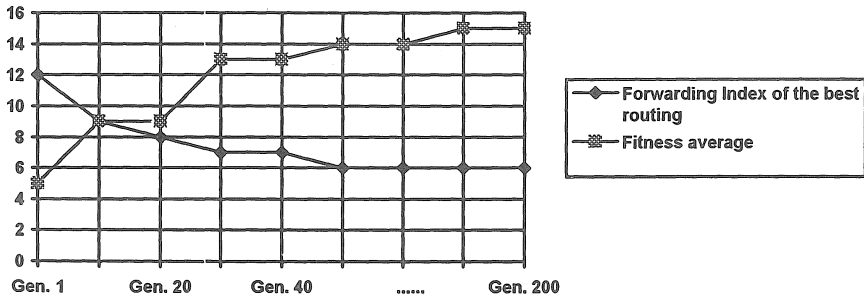
We study the behavior of the algorithm for three graphs, whose forwarding index are known. First, the cycle C_8 of order 8. Second the wheel W_6 of order 7, i.e. the graph obtained from the cycle of 6 vertices v_1, v_2, \dots, v_6 adding a vertex c and the edges $\{(c, v_i) : 1 \leq i \leq 6\}$. The last, the hypercube Q_3 of order 8, the graph whose vertices are the binary words of length 3, i.e. $V(Q_3) = \{(x_1, x_2, x_3) : x_i \in \{0,1\}\}$. Two vertices are adjacent if and only if the vertices differ only in one coordinate. The order of Q_3 is $8 = 2^3$. The forwarding index of C_8 and Q_3 are $\xi(C_8) = 9$, $\xi(Q_3) = 5$ [2], and the forwarding index of the W_6 satisfies $3 \leq \xi(W_6) \leq 4$ [3]. We obtain the following upper bounds, $\mu(C_8) \leq 27$, $\mu(W_6) \leq 6$, $\mu(Q_3) \leq 10$, using routings of the shortest paths. The following graphical representations show the behavior of the algorithm.



The Wheel W_6

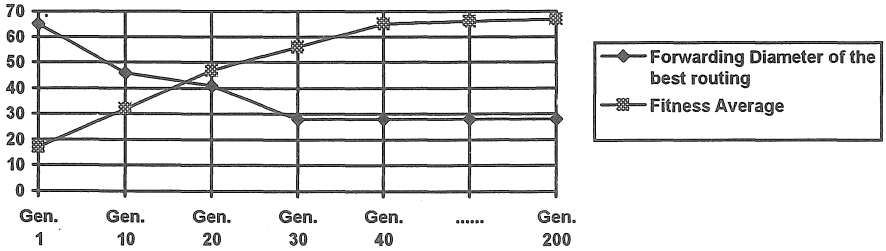


The Hypercube Q_3

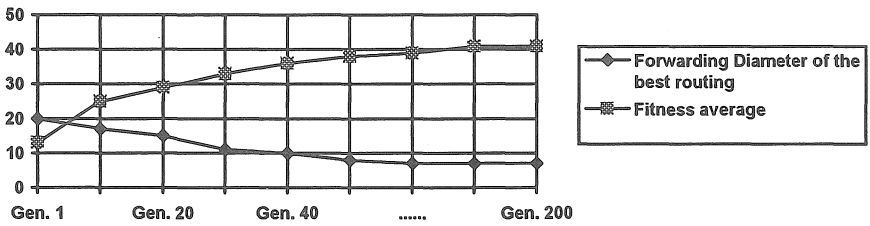


Experimental results for the forwarding diameter

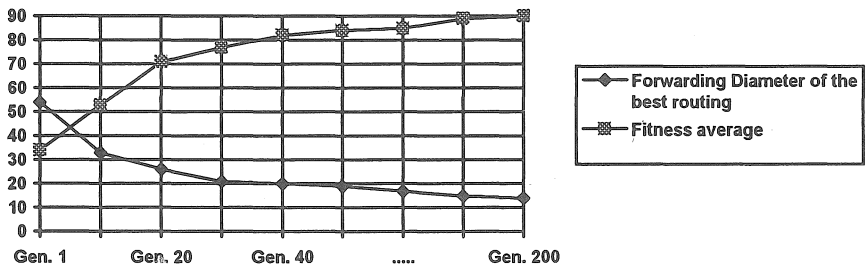
The Cycle C_8



The Wheel W_6



The Hypercube Q_3



Forwarding index	The best value found by the algt.	Forwarding diameter	The best value found by the algt.
$\xi(C_8) = 9$	9	$\mu(C_8) \leq 27$	28
$3 \leq \xi(W_6) \leq 4$	4	$\mu(W_6) \leq 6$	7
$\xi(Q_3) = 5$	6	$\mu(Q_3) \leq 10$	14

Table 1

3 Conclusions

In this paper we have given a genetic algorithm for the forwarding index/diameter problem. The solutions found are optimal, or very close to optimal, or very close to bounds known for the studied graphs (see Table 1). This algorithm can be used also for other kinds of forwarding index/diameter problems, for example forwarding index/diameter by routings of shortest paths or edge-forwarding index [3]. It will be interesting to study the behavior of this algorithm for other kinds of initial populations, for example for a initial population with "many" shortest paths in their routings.

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